

Greedy connectivity of geographically embedded graphs

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We introduce a measure of *greedy connectivity* for geographical networks (graphs embedded in space) and where the search for connecting paths relies only on local information, such as a node's location and that of its neighbors. Constraints of this type are common in everyday life applications. Greedy connectivity accounts also for imperfect transmission across established links and is larger the higher the proportion of nodes that can be reached from other nodes with a high probability. Greedy connectivity can be used as a criterion for optimal network design.

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Large, complex graphs, or networks, have been the subject of much recent interest due to their ubiquity in everyday life and virtually all walks of science [1]. In particular, the ability to connect any two nodes by a continuous path along the graph's edges is crucial to its function (transmitting information, controlling the spread of disease, etc.) and has been studied at length [1].

A graph $G(V,E)$ consists of a set V of N vertices $1, 2, \dots, N$ and a set E of edges, or links (i,j) , connecting between the nodes $(i$ and $j)$. The nodes i and j are *neighbors*. Nodes s and t are *connected* if a continuous path of edges $(s,v_1),(v_1,v_2),\dots,(v_{\ell-1},t)$ can be found between the two nodes. In this view, connectivity is a *global* property: a complete knowledge of the graph is required to decide which pairs of nodes are connected.

In this letter we address the question of connectivity in a different, yet commonly encountered setting. Consider, for example, the paradigmatic experiment of the social psychologist Stanley Milgram [2], who asked people in one part of the USA to deliver a postcard to another person in some faraway location. The name and address of the target person was disclosed, but the participants were to deliver the postcards only to people they knew on a first-name basis. If they did not know the target person, the postcard was to be delivered to an acquaintance, who would then deliver it onward following the same rules, etc. About 20% of the cards reached their target, taking an average of 5.5 steps, a result that gave rise to the idea of “six degrees of separation” and the small-world phenomenon [3].

Two major ingredients are different in Milgram's experiment from the usual concept of graph connectivity: (1) Connectivity is established from *local* information alone—participants knew little else beyond their own acquaintances and had no access to the full net of social contacts. (2) The network in question is embedded in space, i.e., each node (person) has a well defined location. The decision who to mail the postcard to is clearly influenced by distance from

the target. This situation is not uncommon: global information is rarely available in large complex networks, while geographically embedded nets include numerous important examples, such as routers of the Internet, networks of flight connections, the electricity power grid, and neurons in the brain, to name a few, and navigation from local information is an attractive problem [4–13].

Consider then a graph $G(V,E)$ embedded in space, and denote the geographical (Euclidean) distance between nodes by $d(i,j)$. We wish to establish whether a source node s is connected to a target node t , whose location is disclosed, relying only on local information. Inspired by Kleinberg [11,12], we model the search for connectivity by the greedy algorithm: *Make the next step to that neighbor that is closest to the target, provided that the distance diminishes*. Or, symbolically, $(s,v_1,v_2,\dots,v_{\ell-1},t)$ is a *greedy path* of length ℓ from $s \equiv v_0$ to $t \equiv v_\ell$ if for $k=1, 2, \dots, \ell$

$$d(v_k,t) < d(i,t), \quad \text{and} \quad d(v_k,t) < d(v_{k-1},t), \quad (1)$$

for all the neighbors $i \neq v_k$ of v_{k-1} . We are assuming that the nodes are placed in a continuum so that no two pairs of nodes are at the same distance from one another. With this understanding, greedy paths are unique. If Eq. (1) is fulfilled for some ℓ , we say that s and t are *greedily connected*.

By definition, a greedy path is automatically a path. The converse is not true. Many other properties differentiate between connectivity and greedy connectivity: a greedy path is not necessarily reversible—the greedy path found from s to t is not always a greedy path from t to s ; There is no transitivity—if i is greedily connected to j and j is greedily connected to k it does *not* follow that i is greedily connected to k , or in other words, the concatenation of greedy paths is not necessarily a greedy path; If $(s,v_1,v_2,\dots,v_{\ell-1},t)$ is a greedy path from s to t , then (v_i,v_{i+1},\dots,t) is a greedy path (from v_i to t), however, other subpaths are not always greedy paths—e.g., (s,v_1,\dots,v_i) might not be a greedy path from s to v_i ; Perhaps most surprisingly, adding links to an existing network does not necessarily increase greedy connectivity and might actually have the opposite effect.

Due to the irreversibility of greedy paths, one cannot define a greedily connected component of a graph. Instead, we propose measuring greedy connectivity by the quantity

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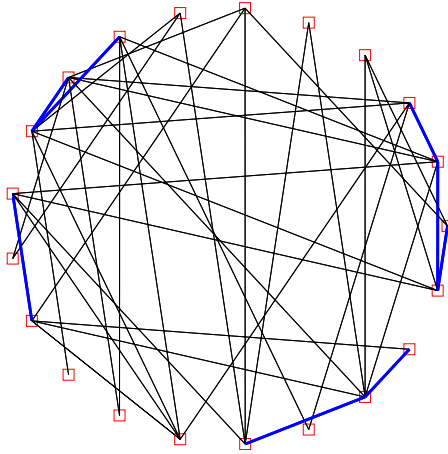


FIG. 1. (Color online) Circle-embedded ER graph with $N=2L+1=21$ nodes and $p=0.2$. “Short-ranged” links, to just one or two nodes away, are highlighted in a different shade.

$$C_G = \sum_{s,t} \sigma_{st} e^{-\mu \ell_{st}} / N(N-1), \quad (2)$$

where the sum runs over all $N(N-1)$ pairs of nodes s and t ($s \neq t$), $\sigma_{st}=1$ if there exists a greedy path from s to t and is 0 otherwise, and ℓ_{st} is the length of the greedy path from s to t (when it exists). For $\mu=0$, C_G is simply the fraction of all pairs that are greedily connected. μ can be thought of as a *chemical potential*, or $\omega \equiv e^{-\mu}$ can be interpreted as the probability to make the transition across a single link successfully. (This is important in situations such as the Milgram experiment, where $\omega < 1$). $C_G(\omega)$ is the actual fraction of successful connections between all possible pairs of nodes when the transmission probability across each link is ω .

We now turn to some key examples. Consider first an Erdős-Rényi (ER) random graph embedded in a circle, where each link is realized with probability p (Fig. 1). For simplicity, assume that the $N=2L+1$ nodes are equally spaced, $\mathbf{r}_j = (\cos \frac{2\pi j}{N}, \sin \frac{2\pi j}{N}) \in \mathbb{R}^2$. To avoid degeneracy of greedy paths, we introduce a small random perturbation to the location of each node. Alternatively, one can work with equal spacing and preserve uniqueness by making an arbitrary random choice when more than one option for a greedy step becomes available. Distances can be measured as either $|\mathbf{r}_i - \mathbf{r}_j|$ or $\min\{|i-j|, N-|i-j|\}$, to the same effect. We opt for the latter.

Denote by $P(\ell, m)$ the probability that two nodes, m lattice spacings apart, are connected by a greedy path of ℓ steps. It obeys the equation

$$P(\ell, m) = pP(\ell-1, 0) + (1-q^2) \sum_{k=1}^{m-1} q^{2k-1} P(\ell-1, k), \quad (3)$$

($q \equiv 1-p$ is the probability that a link is absent), with boundary condition $P(\ell, 0) = \delta_{\ell,0}$. The first term on the rhs denotes the event that there is a direct link between the target and source (probability p) and the boundary condition tells us that the greedy path has then length 1. The first term implied by the sum refers to the case that the direct link is absent (prob. q) but a link to at least one of the two nearest neighbors of the target exists (prob. $1-q^2$); from there, one needs

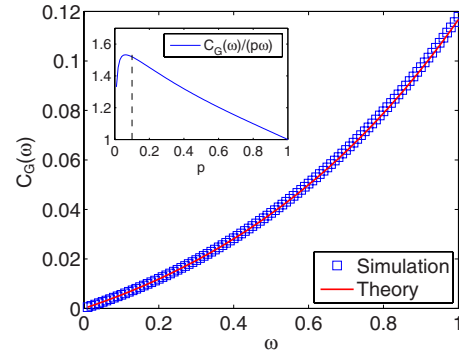


FIG. 2. (Color online) Greedy connectivity of circularly embedded ER graph, with $L=100$ and $p=0.05$, as a function of the transmission probability ω . Inset: The network enhancement factor, $C_G(\omega)/p\omega$, as a function of p , for $\omega=0.5$.

a greedy path of length $\ell-1$ (since one step has already been taken) to the target at distance $k=1$, expressed by the $P(\ell-1, k)$. Successive terms model the events that increasingly more links to the sites surrounding the target are absent.

Equation (3) can be solved in standard ways, to yield

$$P_\omega(m) \equiv \sum_{\ell=1}^m P(\ell, m) \omega^\ell = p\omega \prod_{k=2}^m [1 + \omega(1-q^2)q^{2k-3}]. \quad (4)$$

Finally, using $C_G(\omega) = (1/L) \sum_{m=1}^L P_\omega(m)$, we get

$$C_G(\omega) = p\omega \frac{1}{L} \left(1 + \sum_{m=2}^L \prod_{k=2}^m [1 + \omega(1-q^2)q^{2k-3}] \right). \quad (5)$$

It is interesting to note that $p\omega$ is the greedy connectivity that would result if the only greedy path available between any two nodes were a direct link (that occurs with prob. p). Thus, the remaining factor is the enhancement to the C_G that occurs as a result of other available paths, when the direct link is absent. This enhancement factor is bounded by e^ω and achieves its maximum near $p \sim 1/\sqrt{L}$. Typical results for the greedy connectivity of ER graphs, comparing our theoretical analysis to computer simulations, are shown in Fig. 2.

Next, consider circularly embedded small-world (SW) networks [14]. We start with the underlying “lattice” configuration, where each of the $N=2L+1$ nodes is connected to l -nearest neighbors on either side [Fig. 3(a)]. As before, the nodes are slightly perturbed from their lattice centers, to avoid degeneracy of greedy paths. The equation for $P(\ell, m)$ reads

$$P(\ell, m) = \delta_{\ell, \lfloor m/l \rfloor}, \quad (6)$$

where $\lfloor x \rfloor$ is the smallest integer greater or equal to x . We then have $P_\omega(m) = \sum_{\ell} P(\ell, m) \omega^\ell = \omega^{\lfloor m/l \rfloor}$, and

$$C_G(\omega) = \frac{1}{L} \sum_{m=1}^L P_\omega(m) = \frac{1}{L} (\omega + \omega^2 + \dots + \omega^{L/l}) = p\omega \frac{1 - \omega^{1/p}}{1 - \omega}, \quad (7)$$

where, for simplicity, we have assumed that L is a multiple of l , and we write $l/L=p$ for comparison with ER graphs

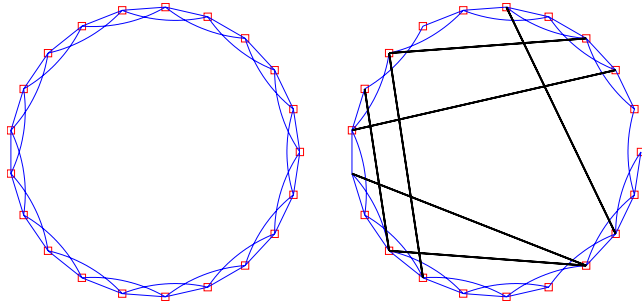


FIG. 3. (Color online) Circularly embedded lattice (left) with $N=2L+1=21$ nodes and $l=2$. Circularly embedded small-world network (right), obtained by removing a fraction $\epsilon=0.2$ of the links and reconnecting them between random pairs of nodes.

(this yields the same number of links in either case). Indeed, it is interesting to note that the greedy connectivity of the lattice is always larger than that of an equivalent ER graph. The lattice architecture guarantees that any two sites are connected, yet the typical distances are order N , rather than $\ln N$, as in ER graphs. The benefits seem to get the upper hand.

To achieve the small-world effect, a fraction ϵ of the links are removed and are then reconnected between randomly selected pairs of nodes (but avoiding multiple connections between any pair), see Fig. 3(b). Even a small fraction ϵ of randomly rerouted links reduces the typical shortest path between nodes, from $O(N)$ to $O(\ln N)$. We now show that the fraction ϵ can be optimized to attain a maximum in the greedy connectivity (in particular, outperforming the lattice, for which $\epsilon=0$).

The equation for the $P(\ell, m)$ of SW networks is

$$\begin{aligned}
 P(\ell, m) = & pP(\ell - 1, 0) + (1 - q^2) \sum_{k=1}^{m-3} q^{2k-1} P(\ell - 1, k) \\
 & + (1 - q'q)q^{2m-5} \{P(\ell - 1, m - 2) \\
 & + q'qP(\ell - 1, m - 1)\}, \tag{8}
 \end{aligned}$$

where we have specialized to the case of $l=2$. For links spanning nodes more than $l=2$ lattice spacings apart the equation is the same as for ER graphs, with $p \equiv \epsilon l/L$, now the effective probability of random long-range links. The only difference is when the first greedy step is to a site within l spacings; these require l specialized terms (the last two terms, in our case) because the probability of such short-range links is $p' \equiv 1 - \epsilon + p$ (and $q' = 1 - p'$), rather than p . Equation (8) is valid for $m \geq 3$. The boundary conditions are revised, for the very same reason,

$$P(\ell, 1) = \delta_{\ell,1} p', \quad P(\ell, 2) = \delta_{\ell,1} p' + \delta_{\ell,2} (1 - q'q) p' q'.$$

Equation (8) can be solved by standard techniques. The final expression we obtain for $C_G(\omega)$ is too cumbersome to list here, but it agrees perfectly well with numerical simulations, as shown in Fig. 4 for one typical case. Note the maximum in C_G , about $\epsilon \approx 0.2$, which is nearly twice as large as the C_G of the corresponding lattice, at $\epsilon=0$, and about seven times as large as the corresponding ER network, at $\epsilon=1$. Qualitatively similar results are obtained for most other link

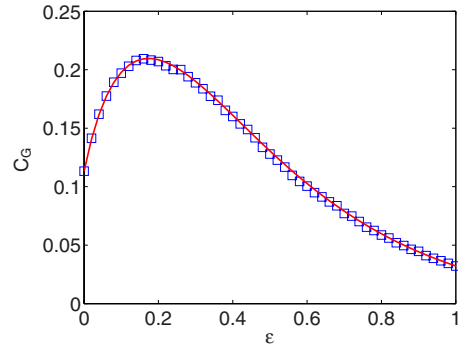


FIG. 4. (Color online) Greedy connectivity of circularly embedded SW graph, with $L=100$, $l=2$, and $\omega=0.85$, as a function of the fraction of random links, ϵ . Theoretical results (solid curve) are compared to numerical simulations (symbols).

densities, $l \geq 2$. For $l=1$, the maximum $C_G(\omega)$ occurs always at $\epsilon=0$, that is, for the underlying lattice (a simple ring).

Our third and last example is that of circularly embedded scale-free (SF) networks. As usual, the $N=2L+1$ nodes are to be placed on a ring, slightly perturbed from their lattice locations, and we start with a single node at $(0,0)$. We then construct a scale-free net according to the *redirection* algorithm of Krapivsky and Redner [15]: each new node is brought in (to a random location) and is connected to one of the existing nodes, selected randomly, with probability $1-r$. With probability r , the connection is redirected to the *ancestor* of that node (the node it was attached to first, when it was added to the net). This yields a graph with scale-free degree distribution, $P(k) \sim k^{-\gamma}$, $\gamma = 1 + 1/r$. Because the networks built in this way are actually trees, the average degree is $\langle k \rangle = 2$, so the procedure has the advantage of keeping a constant density of links even as r , or γ , is varied.

In Fig. 5 we present data culled from computer simulations of circularly embedded SF nets. In the limit of $r \rightarrow 0$ ($\gamma \rightarrow \infty$) the networks are trees with a *narrow* degree distribution, similar to ER graphs. The C_G in that limit is equal to that of equivalent ER graphs (with the same link density). In the opposite limit of $r \rightarrow 1$ ($\gamma \rightarrow 2$) all the links are redirected to one “superhub” and we get a *star graph*. It is easy to show that in this case $C_G = (1/2)\omega^2$ (for $L \geq 1$). As γ decreases

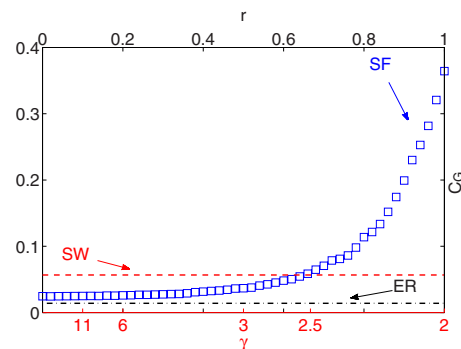


FIG. 5. (Color online) Greedy connectivity of circularly embedded SF nets (with $L=100$ and $\omega=0.85$) as a function of their degree distribution exponent, $\gamma = 1 + 1/r$. The C_G of equivalent ER graphs (— · —) and SW graphs (— —) is shown for comparison.

from ∞ to 2 the C_G of the SF networks increases monotonically, the largest increase occurring between $3 \geq \gamma > 2$, corresponding to the regime encountered in most frequent applications [1]. Note that SF networks with $\gamma \leq 2.5$ exhibit a greater C_G than that of the optimal corresponding SW net (in this case, of $l=1$, a simple ring, or $\epsilon=0$).

In summary, we have introduced a measure of *greedy* connectivity for geographical networks (graphs embedded in space) and where the search for a connecting path might rely only on local information, such as a node's location and that of its neighbors (the ones linked to it). This is useful in a host of situations where the networks are large and complex and global information is not available, or relying on it is impractical due to the network's size. Greedy connectivity is larger the larger the fraction of connected nodes.

Greedy connectivity generalizes the Kleinberg navigation problem (by which it is inspired) in several ways, most importantly, in that nothing is presumed about the network structure; the existence of a greedy path between any two

nodes is *not* required, and the probability of transmission across any given link, ω , and its consequences, is now an integral part of the theory. In particular, Kleinberg-like greedy paths, of minimal length, can be found for any geographically embedded network by maximizing $C_G(\omega)$ in the limit of $\omega \rightarrow 0$ (or $\mu \rightarrow \infty$).

An important feature, suggested by the examples analyzed herein, is that once we take into account imperfect transmission (as represented by ω), greedy connectivity can be enhanced and optimized by varying the network architecture, including the geographical placement of the nodes. This is perhaps the richest venue for future applications.

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